# Bounds for the critical line of the theta -contact processes with 1<or= theta <or=2



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# Bounds for the critical line of the $\theta$-contact processes with $1 \leqslant \theta \leqslant 2$

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**Abstract.** We study a family of the one-dimensional contact processes introduced by Durrett and Griffeath, which is parametrized by $\theta$. For each $\theta \geqslant 1$, there is a unique critical value $\lambda_c(\theta)$ so that any process becomes extinct with probability 1 for $\lambda < \lambda_c(\theta)$, but all processes starting from non-empty initial states have positive probabilities of survival for $\lambda > \lambda_c(\theta)$. In this paper we give rigorous upper and lower bounds for the critical line $\lambda = \lambda_c(\theta)$ for $1 \leqslant \theta \leqslant 2$. In order to obtain the upper bound we extend the Holley–Liggett argument which was originally given for the case $\theta = 2$ (the basic contact process). We construct a new class of attractive renewal measures with positive densities and the upper bound of $\lambda_c(\theta)$ is given as the largest root of a cubic equation, $\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0$. Recently Liggett reported an upper bound of the critical value for the case $\theta = 1$ (the threshold contact process) by a modified version of the Holley–Liggett argument. Our result includes these previous results as special cases.

## 1. Introduction

In the present paper we study a family of one-dimensional contact processes introduced by Durrett and Griffeath (1983). The one-dimensional contact process is a continuous-time Markov process on a lattice $\mathbf{Z}$. The state at time $t$ is given by a set $\eta_t \subset \mathbf{Z}$ of the lattice sites which are occupied by particles. The system evolves as follows:

(i) if $x \in \eta_t$, then $x$ becomes vacant at rate 1,

(ii) if $x \notin \eta_t$, then $x$ becomes occupied at the rate $f(N_x)$ which depends on the number of nearest-neighbour sites $\{x-1, x+1\}$ occupied by particles,

$$N_x = |\eta_t \cap \{x-1, x+1\}| \tag{1.1}$$

where $|A|$ denotes the number of sites in a set $A$. We assume that $f(N)$ is the following function:

$$f(N) = \begin{cases} 0 & \text{if } N = 0 \\ \lambda & \text{if } N = 1 \\ \theta\lambda & \text{if } N = 2 \end{cases} \tag{1.2}$$

where $\lambda$ and $\theta$ are non-negative parameters.

Figure 1 illustrates the above elementary processes. Here the full circles denote particles and the open circles denote vacancies. This process can be viewed as a simple model of

  

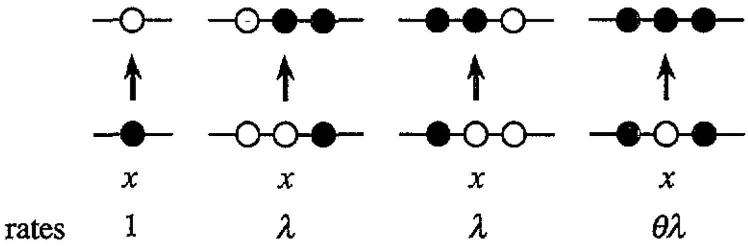


**Figure 1.** The elementary processes of the $\theta$-contact process.

the spread of infection of a disease. An individual at $x \in \mathbf{Z}$ is infected if $x \in \eta_t$ and healthy if $x \notin \eta_t$. The parameter $\lambda$ is the infection rate in the case that one of the neighbours is infected. It is natural to assume that the infection rate in the case where both neighbours are infected is different from that in the case where only one of the neighbours is infected. The parameter $\theta$ is the ratio of the infection rates of these two different situations. The process with $\theta = 2$ is usually called the *basic contact process* (BCP) and was first studied by Harris (1974). If $\theta = 1$, then $f(N) = 0$ for $N = 0$ and $f(N) = \lambda$ for all $N \geqslant 1$. The process with $\theta = 1$ may be called the *threshold contact process* (TCP) (Liggett 1991b).

It is well known that the BCP ($\theta = 2$) is equivalent to the Reggeon quantum spin model in high-energy physics (Brower *et al.* 1978, Grassberger and de la Torre 1979). It should be remarked here that the TCP was also studied in a different context. Dickman and Burschka (1988) introduced a non-equilibrium lattice model as a simplified version of the model for catalytic surfaces proposed by Ziff *et al* (1986). Their model is called the A model but is equivalent to the TCP if the roles of particles and vacancies are exchanged. In this paper we consider a family of the contact processes for

$$1 \leqslant \theta \leqslant 2 . \tag{1.3}$$

We will call this family the $\theta$-contact process (or $\theta$-CP for short) here.

The $\theta$-CP, $\eta_t$, can be defined in the standard method following the textbook of Liggett (1985) for the interacting particle systems as follows. We write $\eta(x) = 1$ if $x \in \eta$ and $\eta(x) = 0$ if $x \notin \eta$. The state space is $X = \{0, 1\}^{\mathbf{Z}}$ and let $C(X)$ be a set of continuous functions on $X$. We define the formal generator $\Omega$ on $C(X)$ as

$$\Omega f(\eta) = \sum_{x \in \mathbf{Z}} c(x, \eta)[f(\eta^x) - f(\eta)] \tag{1.4}$$

for $f \in C(X)$ with

$$c(x, \eta) = \eta(x) + \lambda(1 - \eta(x))\{\eta(x-1) + \eta(x+1) - (2-\theta)\eta(x-1)\eta(x+1)\} \tag{1.5}$$

where $\lambda$ and $\theta$ are non-negative parameters and $\eta^x$ denotes the configuration where $\eta^x(u) = \eta(u)$ for $u \neq x$ and $\eta^x(x) = 1 - \eta(x)$. We consider a spin system in which only one spin flip ($\eta_t(x) = 0 \to 1$ or $\eta_t(x) = 1 \to 0$) can occur in each transition with the flip rate (1.5). The Markov semigroup $S(t)$ is defined by $\Omega$ as $S(t)f = \lim_{n\to\infty}(I - (t/n)\Omega)^{-n} f$ for $f \in C(X)$ and $t \geqslant 0$, where $I$ is the identity operator. There is a unique Markov process $\eta_t$ corresponding to $S(t)$ for each $\lambda$ and $\theta$. The $\theta$-CP can be also constructed by using the graphical representation (Griffeath 1979, Durrett and Griffeath 1983). More detail for the graphical representation of the $\theta$-CP for the case (1.3) is found in Katori and Konno (1993).

If $\theta \geqslant 1$, the addition of an extra particle to the state of the process at some site does not decrease the creation rate of a particle at any vacant site. Of course, it does not increase the annihilation rates of any other particles, because the rate at which an occupied site is vacated is fixed to be 1. Such a property is usually called *attractiveness*. Because the $\theta$-CP is attractive for $\theta \geqslant 1$, the *upper invariant measure* $\nu_{\lambda,\theta}$ exists for each $\lambda \geqslant 0$ and $\theta \geqslant 1$:

$$\nu_{\lambda,\theta} = \lim_{t\to\infty} \delta_1 S(t) \tag{1.6}$$

where $\delta_1$ is a point-mass on $\eta \equiv 1$. In other words, $\nu_{\lambda,\theta}$ is the stationary state of the present process starting from the state with all sites occupied; $\eta_0 = \mathbf{Z}$. Consider the density of particles in this stationary state

$$\rho(\lambda, \theta) = \nu_{\lambda,\theta}\{\eta : \eta(x) = 1\} \tag{1.7}$$

which is independent of $x \in \mathbf{Z}$, since $\nu_{\lambda,\theta}$ is translation-invariant. We can regard $\rho(\lambda, \theta)$ as an order parameter and define a critical value $\lambda_c(\theta)$ by using $\rho(\lambda, \theta)$ for each $\theta \geqslant 1$. We find the following identities:

$$\begin{aligned}\lambda_c(\theta) &\equiv \inf\{\lambda \geqslant 0 : \rho(\lambda, \theta) > 0\}\\ &= \inf\{\lambda \geqslant 0 : P(\eta_t \neq \emptyset\ {}^\forall t \geqslant 0) > 0 \text{ for any non-empty initial state}\}\\ &= \sup\{\lambda \geqslant 0 : P(\eta_t = \emptyset \text{ for some } t \geqslant 0) = 1 \text{ for any initial state}\}.\end{aligned} \tag{1.8}$$

That is, any process becomes extinct with probability 1 if $\lambda < \lambda_c(\theta)$, while if $\lambda > \lambda_c(\theta)$ all processes starting from non-empty states have positive probabilities of survival. If we interpret the $\theta$-CP as a simple model of the spread of infection of a disease, the former may correspond to the extermination of the disease and the latter may represent the state where the infection is spreading and healthy individuals and infected ones coexist. In the context of the chemical reactions, the extinction state may correspond to the poisoned state and the survival state may correspond to the active steady state of the catalytic surfaces. Durrett and Griffeath (1983) observed that the edge speeds characterize the critical values and proved that $\lambda_c(\theta)$ is a strictly decreasing function of $\theta$ if $1 \leqslant \theta \leqslant 2$. When we consider a $(\lambda, \theta)$-plane with $0 \leqslant \lambda < \infty$, $1 \leqslant \theta < \infty$, $\lambda = \lambda_c(\theta)$ gives a critical line which divides two phases of the long-term behaviour of the processes: *extinction phase* ($\lambda < \lambda_c(\theta)$) and *survival phase* ($\lambda > \lambda_c(\theta)$).

Although the long-term behaviour has been well understood (Durrett and Griffeath 1983), the value of $\lambda_c(\theta)$ is still unknown for any $\theta$. For $\theta = 2$ (BCP), there have been many papers giving bounds for the critical value. Among them the bound $\lambda_c(2) \leqslant 2$ given by Holley and Liggett (1978) is the best upper bound, which is much better than the bounds obtained by the contour method (Gray and Griffeath 1982).

In the Holley–Liggett argument, the survival is proved by showing the existence of a renewal measure $\mu$ such that

$$\nu_{\lambda,\theta}\{\eta : \eta(x) = 1 \text{ for some } x \in A\} \geqslant \mu\{\eta : \eta(x) = 1 \text{ for some } x \in A\} \tag{1.9}$$

for any finite subset $A$ of $\mathbf{Z}$ (see Liggett 1985, pp 268–75). Here the renewal measure $\mu$ is defined corresponding to some probability density $f(n)$ of finite mean on $\{1, 2, 3, \ldots\}$ as follows. If $A = \{x_1, x_2, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$, then

$$\begin{aligned}\mu\{\eta : \eta(x_i) = 1 \text{ for } 1 \leqslant i \leqslant n \text{ and } \eta(x) = 0\ {}^\forall x \notin A \text{ such that } x_1 < x < x_n\}\\ = \frac{\prod_{i=1}^{n-1} f(x_{i+1} - x_i)}{\sum_{k=1}^{\infty} k f(k)}.\end{aligned} \tag{1.10}$$

In other words, $\mu$ is the stationary distribution where the distances between successive particles are independent and identically distributed with density $f(n)$. Of course such independence does not hold in the real distribution $\nu_{\lambda,\theta}$. Therefore $\mu$ can be considered an approximation for $\nu_{\lambda,\theta}$. In order to make (1.9) hold for any finite subsets $A$ of $\mathbf{Z}$, the density $f(n)$ should be a logarithmically convex function in $n$. Holley and Liggett derived explicitly such an $f(n)$ which is expressed by using factorials for the BCP ($\theta = 2$) when $\lambda \geqslant 2$. Their result shows that the RHS of (1.9) is positive for all non-empty subsets $A$ when $\lambda \geqslant 2$. Because it follows $\rho(\lambda, 2) > 0$ as a special case $A = \{x\}$, this result implies $\lambda_c(2) \leqslant 2$.

Recently Liggett reported upper bounds for the critical values for two families of contact processes which are different from the $\theta$-CP; the spatially inhomogeneous contact processes (Liggett 1991a), and the periodic threshold contact processes (Liggett 1991b). The latter family contains the TCP as a special case. His result is the following. Let $\lambda_1$ be the largest root of the cubic equation $\lambda^3 - \lambda^2 - 3\lambda + 1 = 0$ ($\lambda_1 = 2.170\ldots$); then $\lambda_c(1) \leqslant \lambda_1$. The proof is also based on the Holley–Liggett argument. However, in this case, the explicit form of $f(n)$ was not given and only the existence of the required renewal measure was shown. Unfortunately, it does not seem very easy to extend this new proof for $\theta \neq 1$.

In the present paper, we extend the original proof of Holley and Liggett (1978) and give the upper bounds for the critical values $\lambda_c(\theta)$ for $1 \leqslant \theta \leqslant 2$ by constructing $f(n)$ explicitly. Our result is the following.

*Theorem 1.1.* Assume that $1 \leqslant \theta \leqslant 2$. Let $\lambda_U(\theta)$ be the largest root of the cubic equation

$$\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2 - \theta)\lambda + (2 - \theta) = 0\,. \tag{1.11}$$

Then $\lambda_c(\theta) \leqslant \lambda_U(\theta)$.

It is easy to confirm that this contains Holley–Liggett's bound $\lambda_c(2) \leqslant 2$ and Liggett's bound $\lambda_c(1) \leqslant \lambda_1$ as special cases. We will derive iterative equations for $f(n)$ and show that $f(n)$ can be expressed by using Gauss's hypergeometric series in the form $F(-(n-2), -(n-1), 2; z)$ for $1 \leqslant \theta \leqslant 2$. In the limit $\theta \to 2$, the variable $z$ becomes 1 and the hypergeometric series is reduced to a combination of the gamma functions (i.e. factorials), reproducing the result of Holley and Liggett (1978) for the BCP ($\theta = 2$). Our result gives another proof of Liggett (1991b) for the TCP ($\theta = 1$).

It is rather easy to give lower bounds for $\lambda_c(\theta)$ if $1 \leqslant \theta \leqslant 2$. The following is a simple extension of the lower bound for the critical value of the BCP, $\lambda_c(2) \geqslant (1 + \sqrt{37})/6 = 1.180\ldots$, given by Griffeath (1975).

*Theorem 1.2.* Let

$$\lambda_L(\theta) = \frac{1}{2(\theta + 1)}\left[\theta - 1 + \sqrt{\theta^2 + 10\theta + 13}\right]. \tag{1.12}$$

Then $\lambda_L(\theta) \leqslant \lambda_c(\theta)$ for $1 \leqslant \theta \leqslant 2$.

The above two theorems give bounds for the critical line $\lambda = \lambda_c(\theta)$ in the phase diagram. Figure 2 shows the numerical values of the upper and lower bounds for the critical line given by our theorems.

By computer simulations or some (non-rigorous) numerical methods, the critical values of the BCP and the TCP are estimated as, $\lambda_c(2) \simeq 1.649$ (Brower *et al* 1978, Konno and

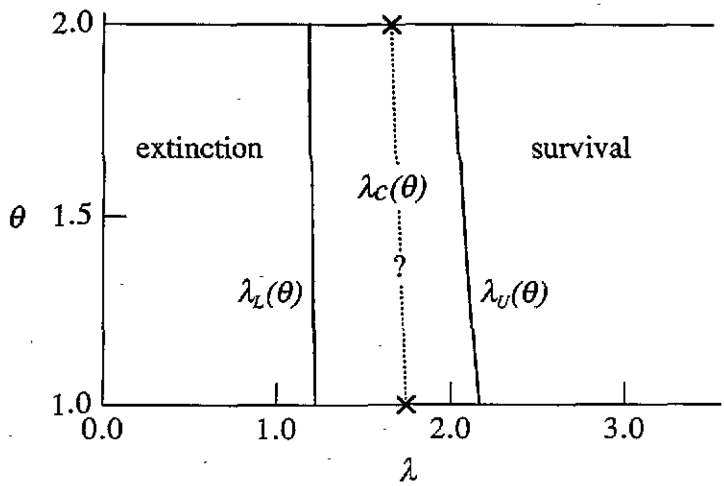


**Figure 2.** Upper and lower bounds for the critical line $\lambda = \lambda_c(\theta)$ given by theorems 1.1 and 1.2 for $1 \leqslant \theta \leqslant 2$. The points marked by × show the values $\lambda_c(1) \simeq 1.742$ and $\lambda_c(2) \simeq 1.649$ estimated by computer simulations or some numerical methods.

Katori 1990) and $\lambda_c(1) \simeq 1.742$ (Dickman and Burschka 1988, Dickman and Jensen 1991, Ferreira and Mendiratta 1993). Notice that the inverse of $T_c = 0.60628 \pm 0.00004$ of the $D = 1$ Reggeon quantum spin model estimated by Brower *et al* (1978) corresponds to $\lambda_c(2)$. Because the TCP is obtained from the A model of Dickman and Burschka (1988) by replacing particles by vacancies and vice versa, $\lambda_c(1)$ is the inverse of the critical value $\lambda_c = 0.574141(2)$ (Dickman and Jensen 1991) of the A model.

The paper is organized as follows. In section 2, we briefly review the Holley–Liggett argument and remark on the required modifications for the present situation. Sections 3 and 4 are devoted to proving theorem 1.1. There we show the probability density $f$ expressed by using the hypergeometric series, which gives the desired renewal measure. The proof of theorem 1.2 is given in section 5. Some concluding remarks are given in section 6.

## 2. The Holley–Liggett argument

The Holley–Liggett argument (1978) treats coalescing dual processes. Let $Y$ be the collection of all finite subsets of $\mathbf{Z}$, and define

$$H(\eta, A) = \prod_{x\in A}[1 - \eta(x)] \tag{2.1}$$

for $\eta \in X, A \in Y$ (the product over the empty set is taken to be 1). It is easy to find that

$$\Omega H(\eta, A) = \sum_{B} q(A, B)[H(\eta, B) - H(\eta, A)] \tag{2.2}$$

where

$$q(A, B) = \sum_{x\in A} c(x) \sum_{C:(A\setminus\{x\})\cup C=B} p(x, C) \tag{2.3}$$

with

$$c(x) = 1 + \theta\lambda \tag{2.4}$$

$$p(x, C) = \begin{cases} 1/(1+\theta\lambda) & \text{if } C = \emptyset \\ (\theta - 1)\lambda/(1+\theta\lambda) & \text{if } C = \{x-1, x\} \text{ or } \{x, x+1\} \\ (2-\theta)\lambda/(1+\theta\lambda) & \text{if } C = \{x-1, x, x+1\} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

where $\Omega$ is the formal generator of the $\theta$-contact processes (1.4). The above defines a continuous-time Markov chain on $Y$ if and only if $1 \leqslant \theta \leqslant 2$, and we write this process as $A_t$. For every $\eta \in X$ and $A \in Y$, the following duality relation holds for any $t \geqslant 0$:

$$S(t)H(\cdot, A)(\eta) = E^A[H(\eta, A_t)] \tag{2.6}$$

where $E^A[\ \cdot\ ]$ denotes the expectation value for the process starting from $A$; $A_0 = A$. This dual process $A_t$ is the same as the coalescing branching processes $\{\hat{\xi}_t^A(\theta, \lambda)\}$ found on page 9 of Durrett and Griffeath (1983). If we regard $A_t$ as a set of the sites occupied by particles, we can say that each particle does three different things. A particle gives birth to a new particle at one of the neighbouring sites at rate $(\theta - 1)\lambda$ for each neighbouring site (*single branching*). At rate $(2-\theta)\lambda$, each particle gives birth to two particles simultaneously at both of the two neighbouring sites (*double branching*), and each particle will be annihilated at rate 1. This process is a coalescing process; if a particle lands on a site which is already occupied, then two particles coalesce to form one particle.

For all $A \in Y$, define

$$\sigma(A) = P^A(A_t \neq \emptyset \text{ for all } t \geqslant 0)‰. \tag{2.7}$$

By the duality relation (2.6), the critical value $\lambda_c(\theta)$ defined by (1.8) is characterized by $\sigma(A)$ for $1 \leqslant \theta \leqslant 2$ as

$$\lambda_c(\theta) = \inf\{\lambda \geqslant 0 : \sigma(A) > 0 \quad {}^\forall A \in Y, A \neq \emptyset\} = \sup\{\lambda \geqslant 0 : \sigma(A) = 0 \quad {}^\forall A \in Y\} \tag{2.8}$$

(see the identity (10) in Durrett and Griffeath 1983).

The Holley–Liggett argument is based on the following proposition as explained on page 268 of Liggett (1985).

*Proposition 2.1.* If

$$h(\emptyset) = 0 \qquad 0 < h(A) \leqslant 1 \qquad \text{for} \qquad A \neq \emptyset \tag{2.9}$$

$$\lim_{|A| \to \infty} h(A) = 1 \tag{2.10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} E^A[h(A_t)]|_{t=0} \geqslant 0 \qquad {}^\forall A \in Y \tag{2.11}$$

then

$$\sigma(A) \geqslant h(A). \tag{2.12}$$

Then the problem is how to choose $h$ which satisfies the assumptions in proposition 2.1. If such an $h$ is chosen and $h(A) > 0$ for all $A \neq \emptyset$ for given $(\lambda, \theta)$, then $\lambda \geqslant \lambda_c(\theta)$ by the identity (2.8). Holley and Liggett proposed the following choice for the BCP ($\theta = 2$).

(a) Choose $h$ of the form

$$h(A) = \mu\{\eta : \eta(x) = 1 \quad \text{for some } x \in A\} \tag{2.13}$$

for some renewal measure $\mu$ on $X$ (with $h(\emptyset) = 0$).

(b) Choose the density $f(n)$ which determines $\mu$ through (1.10) so that (2.11) holds with equality for all $A$ of the form $\{1, 2, \ldots, n\}$.

It should be remarked that, by the duality relation (2.6), (2.7) is written as

$$\sigma(A) = \nu_{\lambda,\theta}\{\eta : \eta(x) = 1 \quad \text{for some } x \in A\} \tag{2.14}$$

by using the upper invariant measure of the $\theta$-CP. Therefore the choice (a) seems to be natural.

If we also take this choice for $\theta \neq 2$, we obtain the following proposition.

*Proposition 2.2.* Let $1 \leqslant \theta \leqslant 2$. Suppose that there are functions $f(n)$ on $\{1, 2, 3, \ldots\}$ which satisfy

$$\begin{aligned} &2\lambda f(3) + f(1)^2 - (2 + \theta\lambda) f(2) = 0 \\ &2\lambda f(n+1) + \sum_{k=1}^{n-1} f(k) f(n-k) - 2(1+\lambda) f(n) = 0 \qquad n \geqslant 3 \end{aligned} \tag{2.15}$$

$$\sum_{n=1}^{\infty} f(n) = 1 \tag{2.16}$$

$$\sum_{n=1}^{\infty} n f(n) < \infty \tag{2.17}$$

$$0 < f(n) < 1 \qquad \forall n \geqslant 1 \tag{2.18}$$

and

$$\frac{f(n)}{f(n+1)} \geqslant \frac{f(n+1)}{f(n+2)} \qquad \forall n \geqslant 1\,. \tag{2.19}$$

Then for all non-empty subsets $A \in Y$

$$\begin{aligned} \sigma(A) &= \nu_{\lambda,\theta}\{\eta : \eta(x) = 1 \quad \text{for some } x \in A\} \\ &\geqslant \mu\{\eta : \eta(x) = 1 \quad \text{for some } x \in A\} > 0 \end{aligned} \tag{2.20}$$

where $\mu$ is the renewal measure corresponding to $f(n)$. It means that the $\theta$-CP survives.

*Proof.* Let $F(n) = \sum_{k=n}^{\infty} f(k)$. From the choice (b), $F(n)$ should satisfy the following equations:

$$\theta\lambda F(2) + (2-\theta)\lambda F(3) = F(1)^2 \tag{2.21}$$

$$\sum_{k=1}^{n} F(k)F(n+1-k) = 2\lambda F(n+1) \qquad n \geqslant 2. \tag{2.22}$$

It should be remarked that (2.16) is written as

$$F(1) = 1. \tag{2.23}$$

It is easy to see that (2.15) with (2.16) is equivalent to (2.21)–(2.23). We can see the detail of the Holley–Liggett argument for $\theta = 2$ on pages 268–75 of Liggett (1985). Because proposition 2.2 is a straightforward extension of a part of lemma 1.25 on page 272 of Liggett (1985) here we only give some remarks on the required modifications for the cases $\theta \neq 2$.

Fix $A \in Y$, and write $A = \cup_{i=1}^{k} A_i$, where $A_i = [l_i + 1, r_i - 1]$ are the ordered maximal connected component of $A$. For $x \in \mathbf{Z}$, define

$$R(x) = \mu\{\eta : \eta = 0 \text{ on } A \cap (x, \infty) | \eta(x) = 1\}$$

and

$$L(x) = \mu\{\eta : \eta = 0 \text{ on } A \cap (-\infty, x) | \eta(x) = 1\}.$$

By the properties of the renewal measure, the inequality (2.11) is written (instead of (1.26) on page 273 of Liggett 1985) as

$$\sum_{x \in A} L(x)R(x) + (2-\theta)\lambda \sum_{i:|A_i|=1} L(l_i)f(2)R(r_i) \leqslant \lambda \sum_{i=1}^{k} [L(l_i)R(l_i) + L(r_i)R(r_i)]. \tag{2.24}$$

The equation (1.28) on page 273 of Liggett (1985),

$$2\lambda f(n) = \sum_{k=1}^{n-1} F(k)f(n-k) - F(n) \tag{2.25}$$

is valid only for $n \geqslant 3$ and the corresponding equation for $n = 2$ is given by

$$2\lambda f(2) = F(1)f(1) - F(2) + (2-\theta)\lambda f(2) \tag{2.26}$$

following (2.21) and (2.22). It should be remarked that if we use (2.25) and (2.26) correctly, we rewrite (2.24) in the following form which is exactly the same as for the case $\theta = 2$:

$$\sum_{x \in A} L(x)R(x) \leqslant \frac{1}{2} \sum_{l=1}^{k} \left[ L(l_i) \sum_{z > l_i, z \in A} F(z - l_i)R(z) + R(r_i) \sum_{z < r_i, z \in A} F(r_i - z)L(z) \right]. \tag{2.27}$$

It is proved by a part of lemma 1.25 on page 272 and lemma 1.24 on page 271 of Liggett (1985) that (2.27) holds if the assumption (2.19) of proposition 2.2 is satisfied. □

## 3. Renewal measures expressed by the hypergeometric series

In order to solve (2.21)–(2.23), we introduce the generating function as done by Holley and Liggett:

$$\phi(u) = \sum_{n=1}^{\infty} F(n) u^n \tag{3.1}$$

where the assumption (2.23) is equivalent to

$$\left.\frac{\mathrm{d}\phi(u)}{\mathrm{d}u}\right|_{u=0} = 1 . \tag{3.2}$$

Let

$$x \equiv \frac{2-\theta}{\lambda} + (\theta - 1) \tag{3.3}$$

$$y \equiv (2-\theta) + \theta\lambda . \tag{3.4}$$

Then we obtain the equation which determines $\phi(u)$ as

$$\phi^2(u) - \frac{2y}{1+x}\phi(u) + \frac{2y}{1+x}u + \frac{1-x}{1+x}u^2 = 0 . \tag{3.5}$$

The unique solution of (3.5) satisfying (3.2) is given by

$$\phi(u) = \frac{y}{1+x}\left[1 - \sqrt{1 - 2\frac{1+x}{y}u - \frac{1-x}{4(1+x)}\left(2\frac{1+x}{y}u\right)^2}\,\right] . \tag{3.6}$$

From now on we will assume that $0 \leqslant x \leqslant 1$ and $y > 0$. Let

$$u_{\pm} = \frac{y}{1 + x \pm \sqrt{2(1+x)}} \tag{3.7}$$

then $u_- < 0 < u_+$. The function (3.6) is real analytic only when $u_- < u < u_+$. It implies that if $u_+ < 1$ there is no real solution $F(n)$ which is summable, since $\sum_{n=1}^{\infty} F(n) = \phi(1)$. On the other hand, if $u_+ \geqslant 1$, that is, if

$$y \geqslant 1 + x + \sqrt{2(1+x)} \tag{3.8}$$

then we can obtain a real solution $F(n)$ of (2.21)–(2.23) by expanding (3.6) in a power series in $u$, which satisfies

$$\sum_{n=1}^{\infty} F(n) = \sum_{n=1}^{\infty} n f(n) = \frac{y}{1+x}\left[1 - \sqrt{1 - 2\frac{1+x}{y} - \frac{1-x^2}{y^2}}\,\right] < \infty . \tag{3.9}$$

To expand (3.6) in a power series in $u$, the following formula is useful.

*Formula 3.1.* When $0 \leqslant x \leqslant 1$ and $0 \leqslant s \leqslant 1$,

$$\sqrt{1 - s - \frac{1-x}{4(1+x)}s^2} = 1 - \sum_{n=1}^{\infty} c_n s^n \tag{3.10}$$

with

$$c_1 = \frac{1}{2} \qquad \text{and} \qquad c_n = \frac{1}{2^{2n-1}}\left(\frac{2}{1+x}\right)^{n/2} \mathrm{e}^{\mathrm{i}(n-2)\varphi} v(n, \mathrm{e}^{-2\mathrm{i}\varphi}) \qquad n \geqslant 2 \tag{3.11}$$

where

$$\mathrm{e}^{\mathrm{i}\varphi} = \sqrt{\frac{1+x}{2}} + \mathrm{i}\sqrt{\frac{1-x}{2}} \tag{3.12}$$

and $v(n, z)$ is Gauss's hypergeometric series in the form

$$v(n, z) = F(-(n-2), -(n-1), 2; z) \qquad n \geqslant 2\,. \tag{3.13}$$

Applying this formula to (3.6), we obtain the following lemma.

*Lemma 3.2.* Let $x = (2-\theta)/\lambda + (\theta - 1)$ and $y = (2-\theta) + \theta\lambda$. If $0 \leqslant x \leqslant 1$ and $y \geqslant 1 + x + \sqrt{2(1+x)}$, then the unique solution of (2.15) which satisfies (2.16) and (2.17) is given by

$$f(n) = F(n) - F(n+1) \tag{3.14}$$

where

$$F(n) = \frac{1}{y^{n-1}} w(n, x)\,. \tag{3.15}$$

Here we define

$$w(1, x) = 1 \qquad \text{and} \qquad w(n, x) = \left(\frac{1+x}{2}\right)^{n/2-1} \mathrm{e}^{\mathrm{i}(n-2)\varphi} v(n, \mathrm{e}^{-2\mathrm{i}\varphi}) \qquad n \geqslant 2 \tag{3.16}$$

$$\mathrm{e}^{\mathrm{i}\varphi} = \sqrt{\frac{1+x}{2}} + \mathrm{i}\sqrt{\frac{1-x}{2}} \tag{3.17}$$

$$v(n, z) = F(-(n-2), -(n-1), 2; z) \qquad n \geqslant 2\,. \tag{3.18}$$

It is remarked here that the hypergeometric series (3.18) satisfies the following iterative equation:

$$(n+2)v(n+2, z) - (2n+1)(1+z)v(n+1, z) + (n-1)(1-z)^2 v(n, z) = 0 \qquad n \geqslant 1$$

with $v(1, z) = v(2, z) = 1$. Then we obtain the following equation for $w(n, x)$, $n \geqslant 1$:

$$(n+2)w(n+2, x) - (2n+1)(1+x)w(n+1, x) - (n-1)(1-x^2)w(n, x) = 0 \tag{3.19}$$

with

$$w(1, x) = w(2, x) = 1 \tag{3.20}$$

where $x$ is defined by (3.3) and $0 \leqslant x \leqslant 1$ is assumed.

Next we will prove the following lemma.

*Lemma 3.3.* If $0 \leqslant x \leqslant 1$ and $y \geqslant 1 + x + \sqrt{2(1+x)}$, $f$ defined by (3.14)–(3.18) satisfies (2.18). In other words, $f$ can be considered as a probability density on $\{1, 2, \ldots\}$ with finite mean.

Since (2.16) is satisfied, what we have to do is to show $f(n) > 0$ $^{\forall}n \geqslant 1$. From (3.14) and (3.15), it is equivalent to

$$y > a_n \qquad {}^{\forall}n \geqslant 1 \tag{3.21}$$

where

$$a_n(x) \equiv \frac{w(n+1, x)}{w(n, x)} \qquad n \geqslant 1\,. \tag{3.22}$$

On the behaviour of the series $\{a_n(x)\}_{n=1,2,\ldots}$, the following lemma will be proved.

*Lemma 3.4.* Let

$$a^*(x) = 1 + x + \sqrt{2(1+x)}\,. \tag{3.23}$$

For $0 \leqslant x \leqslant 1$

$$\lim_{n\to\infty} a_n(x) = a^*(x) \tag{3.24}$$

$$a_1(x) = 1 \qquad \text{and} \qquad a_n(x) < a_{n+1}(x) < a^*(x) \qquad {}^{\forall}n \geqslant 2\,. \tag{3.25}$$

Then if $y \geqslant a^*(x)$, (3.21) holds. Now we prove lemma 3.4.

*Proof of Lemma 3.4.* From (3.19), (3.20) and (3.22), we obtain the iterative equation for $a_n(x)$

$$a_{n+1}(x) = 1 + x + \frac{n-1}{n+2}(1+x)\left\{1 + \frac{1-x}{a_n(x)}\right\} \tag{3.26}$$

with

$$\begin{aligned} &a_1(x) = 1 \qquad a_2(x) = 1 + x \\ &a_3(x) = \frac{1}{2}(3 + 2x) \qquad a_4(x) = \frac{(1+x)(5+2x)}{3+2x}\,. \end{aligned} \tag{3.27}$$

By (3.26),

$$a_n(x) \geqslant 1 + x \qquad {}^{\forall}n \geqslant 2 \qquad \text{if} \qquad 0 \leqslant x \leqslant 1 \tag{3.28}$$

and we can derive from (3.26) the following equation for $n \geqslant 2$:

$$\begin{aligned} a_{n+3}(x) - a_{n+2}(x) &= \frac{(n-1)n}{(n+2)(n+3)} \frac{(1-x^2)^2}{a_n(x)a_{n+1}^2(x)a_{n+2}(x)}(a_{n+1}(x) - a_n(x)) \\ &\quad + \frac{3(1+x)}{n+3}\left\{\frac{1}{n+4} - \frac{n}{(n+2)(n+3)}\frac{1-x^2}{a_{n+1}(x)a_{n+2}(x)}\right\} \\ &\quad + \frac{3}{n+3}\frac{1-x^2}{a_{n+2}(x)}\left\{\frac{1}{n+4} - \frac{n}{(n+2)(n+3)}\frac{1-x^2}{a_{n+1}^2(x)}\right\}. \end{aligned} \tag{3.29}$$

It is easy to confirm using (3.28) that the second term of the right side of (3.29) is positive and its third term is non-negative if $0 \leqslant x \leqslant 1$. Then, for $n \geqslant 2$

$$a_n(x) < a_{n+1}(x) \Longrightarrow a_{n+2}(x) < a_{n+3}(x) \qquad \text{if} \qquad 0 \leqslant x \leqslant 1\,. \tag{3.30}$$

Because

$$a_2(x) < a_3(x) < a_4(x) \qquad \text{if} \qquad x \geqslant 0 \tag{3.31}$$

by (3.27), it is proved that $a_n(x)$ is increasing in $n$ for $n \geqslant 2$, if $0 \leqslant x \leqslant 1$. The limit (3.24) is obtained by taking the limit $n \to \infty$ in (3.26) with (3.28). □

## 4. The monotonicity of $f(n)/f(n+1)$

In this section we will prove the following lemma.

*Lemma 4.1.* If $0 \leqslant x \leqslant 1$ and $y \geqslant a^*(x) = 1+x+\sqrt{2(1+x)}$, $f$ defined by (3.14)–(3.18) satisfies (2.19).

*Remark 4.1.* It is known that the stationary renewal measure with positive density $f$ is a reversible measure for the nearest-particle systems with birth rates

$$\beta(l,r) = \frac{f(l)f(r)}{f(l+r)}. \tag{4.1}$$

The monotonicity of $f(n)/f(n+1)$, (2.19), means that this system is attractive.

By (3.14) and (3.15), $f(n)/f(n+1) \geqslant f(n+1)/f(n+2)$ is equivalent to

$$\frac{1}{a_n(x)}\frac{y-a_n(x)}{y-a_{n+1}(x)} \geqslant \frac{1}{a_{n+1}(x)}\frac{y-a_{n+1}(x)}{y-a_{n+2}(x)} \tag{4.2}$$

where $a_n(x)$ is defined by (3.22). By lemma 3.4, if $y \geqslant a^*(x)$, (4.2) is equivalent to

$$\mathcal{F}_n(y) \geqslant 0 \tag{4.3}$$

for $n \geqslant 2$ with a quadratic

$$\mathcal{F}_n(z) = z^2 - a_{n+1}(x)\frac{a_{n+2}(x)-a_n(x)}{a_{n+1}(x)-a_n(x)}z + a_n(x)a_{n+1}(x)\frac{a_{n+2}(x)-a_{n+1}(x)}{a_{n+1}(x)-a_n(x)}. \tag{4.4}$$

It is easy to see that $\mathcal{F}_n(z)$ has real roots when $0 \leqslant x \leqslant 1$ by lemma 3.4. Let $y_n(x)$ be the larger root of $\mathcal{F}_n(z)$. If $y \geqslant y_n(x)$, then $f(n)/f(n+1) \geqslant f(n+1)/f(n+2)$.

The following lemma will be proved for $y_n(x)$.

*Lemma 4.2.* If $0 \leqslant x \leqslant 1$,

$$y_n(x) \leqslant a^*(x) \tag{4.5}$$

for all $n \geqslant 2$.

At first we notice that lemma 3.4 guarantees that (4.5) follows if $\mathcal{F}_n(a^*(x)) \geqslant 0$ when $0 \leqslant x \leqslant 1$.

For $0 \leqslant x \leqslant 1$, we define by using (3.16)–(3.18)

$$F^*(n) = \frac{1}{a^*(x)^{n-1}}w(n,x) \qquad n \geqslant 1 \tag{4.6}$$

$$f^*(n) = F^*(n) - F^*(n+1). \tag{4.7}$$

In other words, $f^*(n)$ is obtained from (3.14) and (3.15) by letting $y = a^*(x)$. Remark that $f^*(n)$ is a function of $x$. By lemma 3.4, it is found that $\mathcal{F}_n(a^*(x)) \geqslant 0$ is equivalent to

$$\frac{f^*(n)}{f^*(n+1)} \geqslant \frac{f^*(n+1)}{f^*(n+2)} \qquad 0 \leqslant x \leqslant 1. \tag{4.8}$$

Define

$$g(n) \equiv \frac{f^*(n+1)}{f^*(n)} \qquad n \geqslant 1 \tag{4.9}$$

and

$$\kappa = \kappa(x) = 1 + \sqrt{\frac{2}{1+x}}. \tag{4.10}$$

Then we can prove the following lemma.

*Lemma 4.3.*

$$g(n+1)=\frac{(2n+3)-\kappa}{(n+3)\kappa}+\frac{n-1}{n+3}\frac{\kappa-2}{\kappa}\frac{1}{g(n)} \qquad n\geqslant 1 \tag{4.11}$$

with

$$g(2)=\frac{5-\kappa}{4\kappa} \qquad g(3)=\frac{\kappa^2-4\kappa+7}{\kappa(5-\kappa)} \qquad g(4)=-\frac{\kappa^3-9\kappa^2+21\kappa-21}{2\kappa(\kappa^2-4\kappa+7)}. \tag{4.12}$$

For $0\leqslant x\leqslant 1$

$$g(n)\leqslant g(n+1) \qquad {}^{\forall}n\geqslant 2. \tag{4.13}$$

*Proof of Lemma 4.3.* By (3.19) and (3.15), we obtain the equation

$$(n+2)\lambda^2F(n+2)-(2n+1)\lambda F(n+1)-(n-1)\frac{1-x}{1+x}F(n)=0 \qquad n\geqslant 1 \tag{4.14}$$

where we have used (3.3) and (3.4). It is easy to derive the following equation from (4.14):

$$\begin{aligned}(n+3)\lambda^2 f(n+2)&-\{(2n+3)-\lambda\}\lambda f(n+1)-(n-1)\frac{1-x}{1+x}f(n)\\ &=\left(\lambda^2-2\lambda-\frac{1-x}{1+x}\right)F(n+1) \qquad n\geqslant 1.\end{aligned} \tag{4.15}$$

The corresponding equation for $f^*(n)$ is obtained by letting $y=a^*(x)$. Here we remark that, because of (3.3) and (3.4), for each $0\leqslant x\leqslant 1$

$$y=a^*(x)\Longleftrightarrow \lambda=\kappa(x) \tag{4.16}$$

and that

$$\kappa^2(x)-2\kappa(x)-\frac{1-x}{1+x}=0. \tag{4.17}$$

Then

$$(n+3)\kappa f^*(n+2)-\{(2n+3)-\kappa\}f^*(n+1)-(n-1)(\kappa-2)f^*(n)=0 \qquad {}^{\forall}n\geqslant 1.$$

By definition (4.9), we have (4.11). Letting $n=1$ in (4.11), we have $g(2)=(5-\kappa)/4\kappa$, with which we obtain (4.12) by the iterative use of (4.11).

It is easy to derive the following equation from (4.11) for $n\geqslant 2$:

$$\begin{aligned}g(n+3)-g(n+2)&=\frac{(n-1)n}{(n+3)(n+4)}\left(\frac{\kappa-2}{\kappa}\right)^2\frac{1}{g(n)g(n+1)^2g(n+2)}(g(n+1)-g(n))\\ &\quad+\frac{1}{n+4}\frac{3+\kappa}{\kappa}\left\{\frac{1}{n+5}-\frac{n}{(n+3)(n+4)}\frac{\kappa-2}{\kappa}\frac{1}{g(n+1)g(n+2)}\right\}\\ &\quad+\frac{4}{n+4}\frac{\kappa-2}{\kappa}\frac{1}{g(n+2)}\left\{\frac{1}{n+5}-\frac{n}{(n+3)(n+4)}\frac{\kappa-2}{\kappa}\frac{1}{g(n+2)^2}\right\}.\end{aligned} \tag{4.18}$$

Note that the assumption $0 \leqslant x \leqslant 1$ means that

$$2 \leqslant \kappa \leqslant 1+\sqrt{2}. \tag{4.19}$$

By direct calculation using (4.12), we can show that, if (4.19) is satisfied,

$$g(2) \leqslant g(3) \leqslant g(4). \tag{4.20}$$

Now we prove (4.13) by induction. Suppose that

$$g(2) \leqslant g(3) \leqslant \cdots \leqslant g(n) \leqslant g(n+1) \leqslant g(n+2). \tag{4.21}$$

By (4.18) and (4.19), we can conclude that $g(n+2) \leqslant g(n+3)$ from (4.21) if

$$g(n+1)g(n+2) \geqslant \frac{n(n+5)}{(n+3)(n+4)}\frac{\kappa-2}{\kappa}. \tag{4.22}$$

On the other hand, it is easy to show by (4.11) that

$$\begin{aligned} g(n+1)g(n+2) &= \frac{(2n+5)-\kappa}{(n+4)\kappa}g(n+1) + \frac{n}{n+4}\frac{\kappa-2}{\kappa} \\ &\geqslant \frac{(2n+5)-\kappa}{(n+4)\kappa}\frac{(2n+3)-\kappa}{(n+3)\kappa} + \frac{n}{n+4}\frac{\kappa-2}{\kappa} \end{aligned} \tag{4.23}$$

if (4.19) is assumed. By direct computation, we can see that the right side of (4.23) is greater than that of (4.22) for $n \geqslant 1$, if (4.19) is assumed. □

*Proof of Lemma 4.1.* Lemma 4.2 follows (4.13) of lemma 4.3, and as explained just above lemma 4.2, if $0 \leqslant x \leqslant 1$ and $y \geqslant a^*(x)$, then

$$\frac{f(n)}{f(n+1)} \geqslant \frac{f(n+1)}{f(n+2)} \qquad {}^{\forall}n \geqslant 2. \tag{4.24}$$

On the other hand, we can confirm by direct calculation that

$$\frac{f(1)}{f(2)} \geqslant \frac{f(2)}{f(3)} \quad \text{for} \quad y \geqslant a^*(x). \tag{4.25}$$

□

Combining Lemmas 3.2, 3.3 and 4.1 gives the following proposition.

*Proposition 4.4.* Let $x = (2-\theta)/\lambda + (\theta-1)$ and $y = (2-\theta) + \theta\lambda$. If $0 \leqslant x \leqslant 1$ and $y \geqslant 1 + x + \sqrt{2(1+x)}$, then $f$ defined by (3.14)–(3.18) satisfies all of the assumptions (2.15)–(2.19) of proposition 2.2.

It is easy to show from the definitions (3.3) and (3.4) that when $1 \leqslant \theta \leqslant 2$, if $\lambda \geqslant 1$ and

$$\theta\lambda^3 - (3\theta-2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) \geqslant 0 \tag{4.26}$$

then

$$0 \leqslant x \leqslant 1 \qquad \text{and} \qquad y \geqslant 1 + x + \sqrt{2(1+x)}. \tag{4.27}$$

Therefore the proof of theorem 1.1 is completed. □

The numerical values of the largest root $\lambda_{\mathrm{U}}(\theta)$ of the cubic equation (1.11) which is obtained from (4.26) by replacing the inequality by an equality are given in table 1. There the values of $\lambda_{\mathrm{L}}(\theta)$ given by (1.12) are also listed.

**Table 1.** Numerical values of the upper and lower bounds for the critical values $\lambda_c(\theta)$.

| $\theta$ | $\lambda_L(\theta)$ | $\lambda_U(\theta)$ |
|---|---|---|
| 1.0 | 1.224 | 2.171 |
| 1.1 | 1.219 | 2.148 |
| 1.2 | 1.214 | 2.127 |
| 1.3 | 1.209 | 2.107 |
| 1.4 | 1.204 | 2.089 |
| 1.5 | 1.2 | 2.072 |
| 1.6 | 1.195 | 2.056 |
| 1.7 | 1.191 | 2.041 |
| 1.8 | 1.187 | 2.027 |
| 1.9 | 1.184 | 2.013 |
| 2.0 | 1.180 | 2 |

## 5. Proof of Theorem 1.2

By (2.1)–(2.5), we can easily obtain the following identity for any $A \in Y$:

$$\sum_{x\in A}[\sigma(A\setminus\{x\})-\sigma(A)]+(\theta-1)\lambda\sum_{x\in A}\{[\sigma(A\cup\{x-1\})-\sigma(A)]+[\sigma(A\cup\{x+1\})-\sigma(A)]\}$$
$$+(2-\theta)\lambda\sum_{x\in A}[\sigma(A\cup\{x-1,x+1\})-\sigma(A)]=0 \tag{5.1}$$

where $\sigma(A)$ is defined by (2.7) and $\sigma(\emptyset)=0$.

Using (5.1) for all $A\subseteq\{x,x+1,x+2\}$ and by the translation invariance and symmetry of the mechanism, we obtain the following identities.

*Lemma 5.1.* For $x\in\mathbf{Z}$, let $\sigma_1=\sigma(\{x\})$, $\sigma_2=\sigma(\{x,x+1\})$, $\sigma_3=\sigma(\{x,x+1,x+2\})$, $\sigma_4=\sigma(\{x,x+2\})$, $\sigma_5=\sigma(\{x,x+1,x+2,x+3\})$, and $\sigma_6=\sigma(\{x,x+1,x+3\})$. Then

$$\sigma_2=\frac{\theta\lambda+(3-\theta)}{\theta\lambda+(2-\theta)}\sigma_1 \tag{5.2}$$

$$\sigma_3=\frac{\theta\lambda^2+(3-\theta)\lambda+1}{\lambda\{\theta\lambda+(2-\theta)\}}\sigma_1 \tag{5.3}$$

$$2\lambda\sigma_5=-\sigma_4+(2\lambda+3)\sigma_3-2\sigma_2 \tag{5.4}$$

$$(\theta-1)\lambda\sigma_6+(2-\theta)\lambda\sigma_5=(1+\theta\lambda)\sigma_4-(\theta-1)\lambda\sigma_3-\sigma_1\,. \tag{5.5}$$

It should be remarked that proposition 5.9 on page 165 of Liggett (1985) is valid for the present $\theta$-CP. Then $\sigma(A)$ is *submodular* in the sense that

$$\sigma(A\cup B)+\sigma(A\cap B)\leqslant\sigma(A)+\sigma(B) \tag{5.6}$$

whenever $A, B\in Y$.

Using (5.6) for $A=\{x,x+1,x+2\}$ and $B=\{x+1,x+2,x+3\}$, and for $A=\{x,x+1\}$ and $B=\{x+1,x+3\}$, we obtain the following inequalities.

*Lemma 5.2.*

$$\sigma_5 \leqslant 2\sigma_3 - \sigma_2 \tag{5.7}$$

$$\sigma_6 \leqslant \sigma_4 + \sigma_2 - \sigma_1 . \tag{5.8}$$

*Proof of Theorem 1.2.* Combining (5.4), (5.5), (5.7) and (5.8), we obtain for $1 \leqslant \theta \leqslant 2$

$$\sigma_4 \geqslant -(2\lambda - 3)\sigma_3 + 2(\lambda - 1)\sigma_2$$

$$(\lambda + 1)\sigma_4 \leqslant (3 - \theta)\lambda\sigma_3 + (2\theta - 3)\lambda\sigma_2 - [(\theta - 1)\lambda - 1]\sigma_1 .$$

From them the following inequality is derived:

$$\{2\lambda^2 + (2-\theta)\lambda - 3\}\sigma_3 + \{-2\lambda^2 + (2\theta - 3)\lambda + 2\}\sigma_2 - \{(\theta - 1)\lambda - 1\}\sigma_1 \geqslant 0 .$$

By (5.2) and (5.3), it is equivalent to

$$\{(\theta + 1)\lambda^2 - (\theta - 1)\lambda - 3\}\sigma_1 \geqslant 0 \tag{5.9}$$

if $1 \leqslant \theta \leqslant 2$. Therefore, if $\lambda < \lambda_L(\theta)$, where $\lambda_L(\theta)$ is defined by (1.12), $\sigma_1 = 0$. By the submodularity (5.6), $\sigma(A) = 0$ for any $A \in Y$ if $\sigma_1 = 0$. Then by the identity (2.8), theorem 1.2 is proved. □

## 6. Concluding remarks

In this section, we give some comments on our results. First we remark that we can obtain the lower bound for the order parameter $\rho(\lambda, \theta)$, (1.7), as a corollary of propositions 2.2 and 4.4, because $\rho(\lambda, \theta) = \sigma(\{x\})$ by (2.14).

*Corollary 6.1.* Let $\lambda_U(\theta)$ be the upper bound of $\lambda_c(\theta)$ defined in theorem 1.1. For $1 \leqslant \theta \leqslant 2$ and $\lambda \geqslant \lambda_U(\theta)$, $\rho(\lambda, \theta) \geqslant \rho_L(\lambda, \theta)$, where

$$\rho_L(\lambda, \theta) = \frac{\lambda\{\theta\lambda + (2-\theta)\}}{2\theta\lambda^2 + 3(2-\theta)\lambda - (2-\theta)} \times \left\{1 + \frac{1}{\lambda}\sqrt{\frac{\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta)}{\theta\lambda + (2-\theta)}}\right\} . \tag{6.1}$$

It should be remarked that $\rho_L(\lambda, \theta)$ is the inverse of $\phi(1)$ given by (3.9) and that for $1 \leqslant \theta \leqslant 2$

$$\rho_L(\lambda_U(\theta), \theta) = \frac{1}{\lambda_U(\theta)} \tag{6.2}$$

$$\rho_L(\lambda, \theta) > 0 \qquad {}^{\forall}\lambda \geqslant \lambda_U(\theta) \tag{6.3}$$

$$\lim_{\lambda \to \infty} \rho_L(\lambda, \theta) = 1 . \tag{6.4}$$

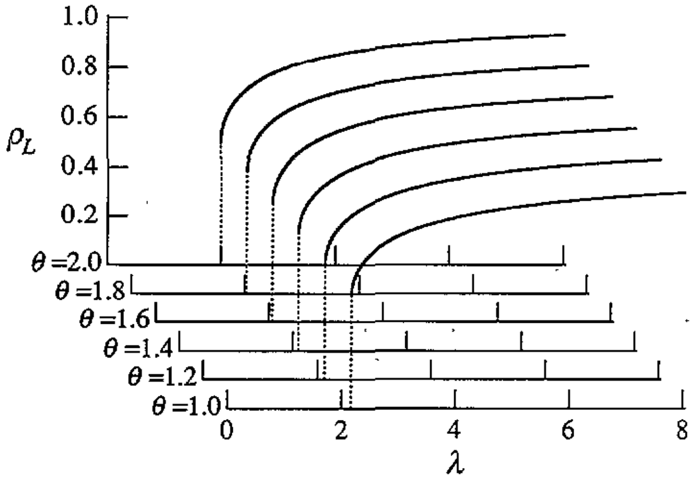


**Figure 3.** Lower bounds $\rho_L(\lambda,\theta)$ for the order parameters $\rho(\lambda,\theta)$ given by corollary 6.1. Notice that these bounds are positive for $\lambda \geqslant \lambda_U(\theta)$, which implies $\lambda_c(\theta) \leqslant \lambda_U(\theta)$.

The lower bounds $\rho_L(\lambda,\theta)$ for the order parameters $\rho(\lambda,\theta)$ given by (6.1) are shown in figure 3 for various values of $\theta$.

Next we discuss the relationship between our renewal measure corresponding to $f$ defined by (3.14)–(3.18) and others. In the limit $\theta \to 2$, $x$ and $y$ become 1 and $2\lambda$, respectively, and $e^{i\varphi} \to 1$ in (3.17). Because

$$F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{6.5}$$

(3.15) becomes

$$F(n) = \frac{(2(n-1))!}{(n-1)!n!}\frac{1}{(2\lambda)^{n-1}} \tag{6.6}$$

in this limit. This is nothing but the function which gives the renewal measure of Holley and Liggett (see (1.18) on page 270 of Liggett 1985). In other words, the renewal measures corresponding to $f$ defined by (3.14)–(3.18) make a new class of renewal measures parametrized by two variables $x$ and $y$, which contains Holley–Liggett's as a special case. In order to obtain an upper bound for the critical value of the TCP, Liggett (1991b) defined $f(t,n)$ for $t \geqslant 0$ and $n \geqslant 1$ as a solution of a set of the differential equations with respect to time $t$ under some initial condition. When $\theta = 1$, the probability density $f(n)$ defined in the present paper may be the limit of this $f(t,n)$ in $t \to \infty$. We hope that the present study will extend the utility of the Holley–Liggett argument for proving the survival of processes.

Recently many kinds of non-equilibrium stochastic lattice models have been introduced and studied intensively to understand non-equilibrium phase transitions (Dickman 1993). Algorithms for computer simulations and series expansion techniques have been developed and the efficiency of these methods are also reported for these non-equilibrium models as well as for equilibrium spin models. However, exact or rigorous results for non-equilibrium models are still few in comparison with equilibrium systems. The work reported here is one of the trials to extend the statements which can be proved rigorously for the non-equilibrium stochastic lattice models.

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